1.)

a.)

The Lagrangian is given by:

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy \tag{1}$$

We can write the x and y components in terms of the angle, θ, and and the driving term:

$$\begin{aligned} y &= l(1-\cos(\theta)) \qquad (2)\\ x &= l\sin(\theta) + x_0\cos(\omega t) \qquad (3) \end{aligned}$$

Taking the time derivate of Eq (1) and (2) and plugging into the Lagrangian:

$$L = \frac{1}{2}\left(l^2\dot{\theta}^2 - 2l\omega x_0\dot{\theta}\cos\theta\sin(\omega t) + \omega^2 x_0^2\sin^2(\omega t)\right) - mgl(1 - cos(\theta))$$

Keeping only terms to the first power of ω, we can find the equation of motion for θ:

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta} \\ \frac{d}{dt}\left[ml^2\dot{\theta} - ml\omega x_0\cos(\theta)\sin(\omega t)\right] = -mgl\sin(\theta) + m\omega l x_0\dot{\theta}\sin(\theta)\sin(\omega t) \\ ml^2\ddot{\theta} + \cancel{ml\omega x_0\dot{\theta}\sin(\theta)\sin(\omega t)} - ml\omega^2 x_0\cos(\theta)\cos(\omega t) = -mgl\sin(\theta) + \cancel{m\omega l x_0\dot{\theta}\sin(\theta)\sin(\omega t)} \\ ml^2\ddot{\theta} = -mgl\sin(\theta) + ml\omega^2 x_0\cos(\theta)\cos(\omega t) \\ \ddot{\theta} = -\frac{g}{l}\sin(\theta) + \frac{\omega^2 x_0}{l}\cos(\theta)\cos(\omega t) \qquad (4) \end{gathered}$$

We now assume that the motion, $\theta(t)$, can be separated into fast and slow components: $\theta = \bar{\theta} + \tilde{\Theta}$. I will use the notation that $\bar{\theta}$ corresponds to the slow motion while $\tilde{\Theta}$ corresponds to the fast motion. We now plug our definition of θ into Eq. (4) and expand in powers of $\tilde{\Theta}$:

$$\begin{aligned} \ddot{\bar{\theta}} + \ddot{\tilde{\Theta}} &= -\frac{g}{l}\sin(\bar{\theta} + \tilde{\Theta}) + \frac{\omega^2 x_0}{l}\cos(\bar{\theta} + \tilde{\Theta})\cos(\omega t) \\ &= \underbrace{-\frac{g}{l}\sin(\bar{\theta})}_{1} - \underbrace{\frac{g}{l}\tilde{\Theta}\cos(\bar{\theta})}_{2} + \underbrace{\frac{\omega^2 x_0}{l}\cos(\bar{\theta})\cos(\omega t)}_{3} - \underbrace{\frac{\omega^2 x_0}{l}\tilde{\Theta}\sin(\bar{\theta})\cos(\omega t)}_{4} \end{aligned}$$

We can now average over the fast period. By doing so, all of the "fast" terms in the above equation will be approximately zero. We can see that term 1 is a slow term as well as term 4. Term 4 is slow due to the beat phenomenon, which we will see from the multiplication of $\tilde{\Theta}$ and $\cos(\omega t)$ in this term. Terms 2 and 3 are fast terms and will go to zero under averaging.

$$< \ddot{\bar{\theta}} >= -\frac{g}{l} < \sin(\bar{\theta}) > -\frac{\omega^2 x_0}{l} < \tilde{\Theta}\cos(\omega t) > \sin(\bar{\theta}) \tag{5}$$

Switching over to the fast components:

$$\ddot{\tilde{\Theta}} = \cancel{-\frac{g}{l}\tilde{\Theta}\cos(\bar{\theta})} + \frac{\omega^2 x_0}{l}\cos(\bar{\theta})\cos(\omega t)$$

The first term went to zero as $\Omega^2 << \omega^2$, where Ω is just the natural freq of the oscillator, $\sqrt{g/l}$ We can integrate this equation with respect to time twice, arriving at:

$$\tilde{\Theta} = -\frac{x_0}{l}\cos(\bar{\theta})\cos(\omega t) \tag{6}$$

We plug this expression for $\tilde{\Theta}$ into Eq (5) to see:

$$< \tilde{\Theta}\cos(\omega t) >= -\frac{\omega^2 x_0}{l} < -\frac{x_0}{l}\cos(\bar{\theta})\cos(\omega t)\cos(\omega t) > \sin(\bar{\theta})$$

$$< \tilde{\Theta}\cos(\omega t) >= +\frac{\omega^2 x_0^2}{l^2} < \cos^2(\omega t) > \cos(\bar{\theta})\sin(\bar{\theta}) \tag{7}$$

The average of $\cos^2(\omega t)$ can be computed:

$$< \cos^2(\omega t) >= \frac{1}{T}\int_0^T \left(\frac{1}{2} + \frac{1}{2}\cos(2\omega t)\right) \mathrm{dt} = \frac{1}{2}$$

And so:

$$< \ddot{\bar{\theta}} >= \ddot{\bar{\theta}} = -\frac{g}{l}\sin(\bar{\theta}) + \frac{\omega^2 x_0^2}{2l^2}\cos(\bar{\theta})\sin(\bar{\theta}) \tag{8}$$

We can write Eq (8) in the form: $\ddot{x} = -\frac{d}{dx}(U)$, where U is the potential:

$$\ddot{\bar{\theta}} = -\frac{d}{d\bar{\theta}}\left[-\frac{g}{l}\cos(\bar{\theta}) - \frac{x_0^2\omega^2}{4l^2}\sin^2\bar{\theta}\right] \tag{9}$$

Eq (9) tells us then there the effective potential is:

$$U_{\text{eff}} = -\frac{g}{l}\cos(\bar{\theta}) - \frac{x_0^2\omega^2}{4l^2}\sin^2\bar{\theta} \tag{10}$$

Our goal is to find the extrema of this function Using a trig identity to reduce the power of $\sin^2(\bar{\theta})$ term and taking a spatial derivative:

$$\frac{dU_{\text{eff}}}{d\bar{\theta}} = \frac{g}{l}\sin(\bar{\theta}) - \frac{x_0^2\omega^2}{4l^2}\sin(2\bar{\theta}) = 0$$

$$\frac{dU_{\text{eff}}}{d\bar{\theta}} = \left(\frac{g}{l} - \frac{x_0^2\omega^2}{2l^2}\cos(\bar{\theta})\right)\sin(\bar{\theta})$$

$$\bar{\theta} = 0 \text{ or } \pi \text{ or } \arccos\left(\frac{2gl}{x_0^2\omega^2}\right)$$

The stability of these extrema are found by taking another spacial derivative.

$$\frac{d^2U_{\text{eff}}}{d\bar{\theta}^2} = \frac{g}{l}\cos(\bar{\theta}) - \frac{x_0^2\omega^2}{2l^2}\cos(2\bar{\theta})$$

For $\bar{\theta} = 0$

$$\left.\frac{d^2U_{\text{eff}}}{d\bar{\theta}^2}\right|_{\bar{\theta}=0} = \frac{g}{l} - \frac{x_0^2\omega^2}{2l^2} \tag{11}$$

Which is positive, and thus stable, if $\frac{g}{l} > \frac{x_0^2\omega^2}{2l^2}$.

For $\bar{\theta} = \pi$

$$\left.\frac{d^2U_{\text{eff}}}{d\bar{\theta}^2}\right|_{\bar{\theta}=\pi} = -\frac{g}{l} - \frac{x_0^2\omega^2}{2l^2} \tag{12}$$

Which is always negative, and thus always unstable.

For $\bar{\theta} = \arccos\left(\frac{2gl}{x_0^2\omega^2}\right)$:

$$\frac{d^2U_{\text{eff}}}{d\bar{\theta}^2} = \frac{g}{l}\cos(\bar{\theta}) - \frac{x_0^2\omega^2}{2l^2}\left[2cos^2(\bar{\theta}) - 1\right]$$

and evaluating at this point:

$$\left.\frac{d^2U_{\text{eff}}}{d\bar{\theta}^2}\right|_{\bar{\theta}} = \frac{g}{l}\left(\frac{2gl}{x_0\omega^2}\right) - \frac{x_0^2\omega^2}{2l^2}\left[2\left(\frac{2gl}{x_0^2\omega^2}\right)^2 - 1\right]$$
$$\left.\frac{d^2U_{\text{eff}}}{d\bar{\theta}^2}\right|_{\bar{\theta}} = -\frac{2g^2}{x_0^2\omega^2} + \frac{x_0^2\omega^2}{2l^2}$$

Which is stable if:

$$\frac{x_0^2\omega^2}{2l^2} > \frac{2g^2}{x_0^2\omega^2}$$

Thus we finally have:

$$\boxed{\bar{\theta} = 0 \text{ is a stable equilibrium point if } \frac{g}{l} > \frac{x_0^2\omega^2}{2l^2}}$$

and

$$\boxed{\bar{\theta} = \arccos\left(\frac{2gl}{x_0^2\omega^2}\right) \text{ is a stable equilibrium point if } \frac{x_0^2\omega^2}{2l^2} > \frac{2g^2}{x_0^2\omega^2}}$$

While $\bar{\theta} = \pi$ is always unstable.

b.)

We can write the x and y components as:

$$\begin{aligned} x &= l\sin(\theta) + r_0\cos(\omega t) \\ y &= l\cos(\theta) + r_0\sin(\omega t) \end{aligned}$$

Plugging this into our usual Lagrangian for a pendulum:

$$\begin{aligned} L &= \frac{1}{2}m\left[(l\dot{\theta}\cos(\theta) - r_0\omega\sin(\omega t))^2 + (r_0\omega\cos(\omega t) - l\sin(\theta)\dot{\theta})^2\right] - mgl(1-\cos(\theta) \\ L &= \frac{1}{2}m\left[l^2\dot{\theta}^2 + r_0^2\omega^2 - 2r_0\omega l\dot{\theta}\sin(\omega t)\cos(\theta) - 2r_0\omega l\dot{\theta}\cos(\omega t)\sin(\theta)\right] - mgl(1-\cos(\theta)) \end{aligned}$$

We can now find the equations of motion:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{\theta}}\right] = \frac{\partial L}{\partial\theta}$$

$$\begin{aligned} ml^2\ddot{\theta} - r_0\omega lm \quad &\Big(\omega\cos(\omega t)\cos(\theta) - \dot{\theta}\sin(\omega t)\sin(\theta) - \omega\sin(\omega t)\sin(\theta) + \dot{\theta}\cos(\omega t)\cos(\theta)\Big) \\ &= -mgl\sin(\theta) + r_0\omega l\dot{\theta}m\left(\sin(\omega t)\sin(\theta) - \cos(\omega t)\cos(\theta)\right) \end{aligned} \tag{13}$$

We keep only to first order in ω, dropping all higher order terms. Several terms cancel, are we are left with:

$$\begin{aligned} ml^2\ddot{\theta} &= -mgl\sin(\theta) + r_0\omega^2 lm(\cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta)) \\ \ddot{\theta} &= -\frac{g}{l}\sin(\theta) + \frac{r_0\omega^2}{l}\left(\cos(\omega t)\cos(\theta) - \sin(\omega t)\sin(\theta)\right) \end{aligned} \tag{14}$$

As in the previous problem, we assume we can write $\theta = \bar{\theta} + \tilde{\Theta}$ where $\bar{\theta}$ is the slow component and $\tilde{\Theta}$ is the fast component. Plugging this into Eq (14) and expanding in powers of $\tilde{\Theta}$:

$$\ddot{\bar{\theta}} + \ddot{\tilde{\Theta}} = -\frac{g}{l}(\sin(\bar{\theta}) - \tilde{\Theta}\cos(\bar{\theta})) + \frac{r_0\omega^2}{l}\Big((\cos(\bar{\theta}) - \tilde{\Theta}\sin(\bar{\theta}))\cos(\omega t) - (\sin(\bar{\theta}) + \tilde{\Theta}\cos(\bar{\theta}))\sin(\omega t)\Big)$$

We can average over the fast time scale, as before, which will leave only the slow components. The fast components will be averaged to zero:

$$\ddot{\bar{\theta}} = -\frac{g}{l}\sin(\bar{\theta}) - \frac{r_0\omega^2}{l}\Big(< \tilde{\Theta}\cos(\omega t) > \sin(\bar{\theta}) + < \tilde{\Theta}\sin(\omega t) > \cos(\bar{\theta})\Big) \tag{15}$$

Switching over to the fast components:

$$\ddot{\tilde{\Theta}} = \cancel{-\frac{g}{l}\tilde{\Theta}\cos(\bar{\theta})} + \frac{r_0\omega^2}{l}\left(\cos(\bar{\theta})\cos(\omega t) - \sin(\bar{\theta})\sin(\omega t)\right) \tag{16}$$

We can easily integrate Eq (16) and find an expression for $\tilde{\Theta}$:

$$\tilde{\Theta} = -\frac{r_0}{l}\big(\cos(\bar{\theta})\cos(\omega t) - \sin(\bar{\theta})\sin(\omega t)\big) \tag{17}$$

Plugging in this expression into the Eq (15) we can calculate the <> terms. To save some computation, we know that any averaging term proportional to $\sin(\omega t)\cos(\omega t)$ will average to zero. Likewise, as seen above in part a), any term proportional to $\sin^2(\omega t)$ or $\cos^2(\omega t)$ will average to 1/2. Simplifying, we arrive at:

$$\ddot{\bar{\theta}} = -\frac{g}{l}\sin(\bar{\theta}) - \frac{r_0^2\omega^2}{l^2}\left(\frac{1}{2}\sin(\bar{\theta})\cos(\bar{\theta}) - \frac{1}{2}\sin(\bar{\theta})\cos(\bar{\theta})\right)$$

$$\ddot{\bar{\theta}} = -\frac{g}{l}\sin(\bar{\theta}) \tag{18}$$

Our effective potential is thus:

$$U_{\text{eff}} = -\frac{g}{l}\cos(\bar{\theta})$$

and the system behaves as if there is NO motion of the pivot point. This gives us the final result of the equilibrium position:

$$\frac{dU_{\text{eff}}}{d\bar{\theta}} = \frac{g}{l}\sin(\bar{\theta}) = 0 \text{ when } \bar{\theta} = 0 \text{ or } \pi$$

$$\frac{d^2U_{\text{eff}}}{d\bar{\theta}^2} = \frac{g}{l}\cos(\bar{\theta})\bigg|_{\bar{\theta}=0} > 0 \text{ always}$$

$$\frac{d^2U_{\text{eff}}}{d\bar{\theta}^2} = \frac{g}{l}\cos(\bar{\theta})\bigg|_{\bar{\theta}=\pi} < 0 \text{ always}$$

$$\boxed{\bar{\theta} = 0 \text{ is a stable equilibrium point.}}$$

$$\boxed{\bar{\theta} = \pi \text{ is an unstable equilibrium point.}}$$

2.)

We are given that the support is driven with $y(t) = y_0\cos(\omega t)$. We begin by finding the Lagrangian:

$$L = \frac{1}{2m}(\dot{x}^2 + \dot{y}^2) - mgy$$

Writing x and y in terms of the angle θ of the pendulum:

$$x = l\sin(\theta)$$

$$y = l\cos(\theta) + y_0\cos(\omega t)$$

We can plug this into the Lagrangian to get:

$$L = \frac{1}{2m}\left[l^2\dot{\theta}^2 + 2ly_0\omega\dot{\theta}\sin(\theta)\sin(\omega t)\right] - mgl(1 - cos(\theta))$$

The equations of motion are therefore:

$$\cancel{m}l^2\ddot{\theta} + \cancel{m}ly_0\omega^2\sin(\theta)\cos(\omega t) + \cancel{mly_0\omega\dot{\theta}\cos(\theta)\sin(\omega t)} = -\cancel{m}gl\sin(\theta) + \cancel{mly_0\omega\dot{\theta}\cos(\theta)\sin(\omega t)}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta + \frac{y_0\omega^2}{l}\sin(\theta)\cos(\omega t) = 0$$

If we set $\omega_0^2 = g/l$:

$$\ddot{\theta} + \omega_0^2\left[1 + \frac{y_0\omega^2}{g}\cos(\omega t)\right]\sin(\theta) = 0$$

We are given that the driving frequency, $\omega = 2\omega_0 + \epsilon$. Thus we can write the above equation as:

$$\ddot{\theta} + \omega_0^2\left[1 + \frac{4y_0}{l}\cos(\omega t)\right]\sin\theta$$

We have used the fact that $\omega^2 = (2\omega_0 + \epsilon)^2 = 4\omega_0^2 + 4\omega_0\epsilon + \epsilon^2 \approx 4\omega_0^2$ to first order in ϵ. A final approximation, the small angle approximation, and letting $h = 4y_0/l$ gives us the final equation:

$$\ddot{\theta} + \omega_0^2\left[1 + h\cos(\omega t)\right]\theta = 0 \tag{19}$$

We can approach this problem but first finding solutions when h=0. This is nothing but the S.H.O, with solution:

$$\theta(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t)$$

When we let $h \neq 0$, we expect the same general form of the solution except for a slow time scale variation of the coefficients.

$$\theta(t) = a(t)\cos((\omega_0 + \frac{\epsilon}{2})t) + b(t)\sin((\omega_0 + \frac{\epsilon}{2})) \tag{20}$$

Plugging this equation for θ into Eq (19), and using $(\omega_0 + \frac{\epsilon}{2}) = \beta$:

$$\begin{gathered}\ddot{a}\cos(\beta t) - 2\beta\dot{a}\sin(\beta t) - \beta^2 a\cos(\beta t) + \ddot{b}\sin(\beta t) + 2\dot{b}\beta\cos(\beta t) - b\beta^2\sin(\beta t)\\ = -\omega_0^2\left[1 + h\cos(\omega t)\right](a\cos(\beta t) + b\sin(\beta t))\end{gathered}$$

Dropping the terms proportional to $\ddot{a}$ and $\ddot{b}$:

$$\begin{gathered}-\beta^2 a\cos(\beta t) - 2\beta\dot{a}\sin(\beta t) + 2\dot{b}\beta\cos(\beta t) - b\beta^2\sin(\beta t)\\ = -\omega_0^2\left[1 + h\cos(\omega t)\right](a\cos(\beta t) + b\sin(\beta t))\end{gathered}$$

We now notice that $\beta^2 = \omega_0^2 + \omega_0\epsilon + \mathcal{O}(\epsilon^2)$:

$$\begin{aligned}-(\cancel{\omega_0^2} + \omega_0\epsilon)a\cos(\beta t) - 2\beta\dot{a}\sin(\beta t) + 2\dot{b}\beta\cos(\beta t) - b(\cancel{\omega_0^2} + \omega_0\epsilon)\sin(\beta t)\\ = -\omega_0^2\left[\cancel{1} + h\cos(\omega t)\right](a\cos(\beta t) + b\sin(\beta t))\end{aligned}$$

Or after some rearranging:

$$\begin{gathered}-\omega_0\epsilon\left(a\cos(\beta t) + b\sin(\beta t)\right) - 2\beta\left(\dot{a}\sin(\beta t) - \dot{b}\cos(\beta t)\right)\\ +\omega_0^2 h\left[a\cos((2\omega_0 + \epsilon)t)\cos(\omega_0 + \frac{\epsilon}{2}) + b\cos((2\omega_0 + \epsilon)t)\sin(\omega_0 + \frac{\epsilon}{2})\right] = 0\end{gathered}$$

We use some trig identities to rewrite the terms in brackets in the above equation:

$$\begin{aligned}\cos((2\omega_0 + \epsilon)t)\cos(\omega_0 + \frac{\epsilon}{2}) &= \frac{1}{2}\cos((\omega_0 + \frac{\epsilon}{2})t) - \frac{1}{2}\cos((3\omega_0 + \frac{3\epsilon}{2})t)\\ \cos((2\omega_0 + \epsilon)t)\sin(\omega_0 + \frac{\epsilon}{2}) &= -\frac{1}{2}\sin((\omega_0 + \frac{\epsilon}{2})t) + \frac{1}{2}\sin((3\omega_0 + \frac{3\epsilon}{2})t)\end{aligned}$$

We only keep terms on resonance, dropping the third harmonic terms, which allows us to write:

$$\begin{aligned}-\omega_0\epsilon\left(a\cos(\beta t) + b\sin(\beta t)\right) - 2\beta\left(\dot{a}\sin(\beta t) - \dot{b}\cos(\beta t)\right)\\ +\omega_0^2 h\left[a\cos(\beta t) - b\sin(\beta t)\right] = 0 \qquad (21)\end{aligned}$$

Or after re-arranging:

$$\left[-\omega_0\epsilon a + 2\beta\dot{b} + \frac{\omega_0^2 h}{2}a\right]\cos(\beta t) + \left[-\omega_0\epsilon b - 2\beta\dot{a} - \frac{\omega_0^2 h}{2}b\right]\sin(\beta t) = 0 \qquad (22)$$

Now Eq (22) has a non-trivial solution if

$$\begin{gathered}-\omega_0\epsilon a + 2\beta\dot{b} + \frac{\omega_0^2 h}{2}a = 0\\ \omega_0\epsilon b - 2\beta\dot{a} - \frac{\omega_0^2 h}{2}b = 0\end{gathered}$$

Or

$$\begin{gathered}\dot{b} - \frac{\epsilon}{2}a + \frac{\omega_0 h}{4}a = 0\\ \dot{a} + \frac{\epsilon}{2}b + \frac{\omega_0 h}{4}b = 0\end{gathered}$$

We assume a solution for $a(t) = a_0\exp(st)$ and $b(t) = b_0\exp(st)$. Now plug into the above equations:

$$sb_0 = \left(\frac{\epsilon}{2} - \frac{\omega_0 h}{4}\right)a_0 \qquad (23)$$

$$sa_0 = -\left(\frac{\epsilon}{2} + \frac{\omega_0 h}{4}\right)b_0 \qquad (24)$$

Multiplying Eq (23) and (24) together:

$$s^2 a_0 b_0 = (-\frac{\epsilon^2}{4} + \frac{\omega_0^2 h^2}{16}) a_0 b_0$$
$$s^2 = \frac{\omega_0^2 h^2}{16} - \frac{\epsilon^2}{4} \tag{25}$$

The growth rate is thus:

$$\boxed{s = \sqrt{\frac{\omega_0^2 h^2}{16} - \frac{\epsilon^2}{4}}}$$

For stable motion, we want s^2 to be negative. For then, s is imaginary, and the coefficients are bounded.

$$\boxed{\text{Stable for } \epsilon^2 > \frac{\omega_0^2 h^2}{4}}$$

Conversely, instability arises if s^2 is positive. For then s is real, and the coefficients grow exponentially.

$$\boxed{\text{Unstable for } \epsilon^2 < \frac{\omega_0^2 h^2}{4}}$$

Winter 2015 **PHYS 200B HW Problem 01.03** Logan Howe

Problem: Compute the threshold for parametric instability in the presence of linear frictional damping, as well as mismatch. For what range of mismatch ϵ will instability occur?

Solution: We should start by writing down Mathieu's equation where linear frictional damping means we have a term proportional to $\dot{\phi}$, i.e.

$$\ddot{\phi} + \gamma\dot{\phi} + \omega_0^2\phi\,(1 + h\cos 2\omega t) = 0 \tag{1}$$

where $\omega = \omega_0 + \epsilon/2$ is half the forcing frequency that results in the parametric resonance and $h = 4y_0/\ell$. A crucial brainwave that we must have is that "threshold" of instability means that instead of posing periodic solutions of the form

$$\phi(t) = a(t)\cos\omega t + b(t)\sin\omega t \tag{2}$$

where the coefficients depend on time and are allowed to blow up, we must instead set the coefficients to constants: $a(t) = a$ and $b(t) = b$. Now proceed to grind by plugging in $\phi(t) = a\cos(\omega t) + b\sin(\omega t)$ into eqaution (1).

$$\ddot{\phi} = -a\omega^2\cos\omega t - b\omega^2\sin\omega t$$

$$\dot{\phi} = \omega\,(-a\sin\omega t + b\cos\omega t)$$

$$\begin{aligned}\implies a\omega^2\cos\omega t - b\omega^2\sin\omega t + \gamma\omega\,(-a\sin\omega t + b\cos\omega t)&\\ + a\omega_0^2\cos\omega t + b\omega_0^2\sin\omega t + \omega_0^2 h\cos 2\omega t\,\big(\cos\omega t + b\omega_0^2 h\sin\omega t\big) &= 0.\end{aligned}$$

Use the trig identity

$$\cos 2\omega t\cos\omega t = \frac{1}{2}\,(\cos\omega t + \cos 3\omega t)$$

to separate into an on-resonance ($\cos\omega t$) and off-resonance ($\cos 3\omega t$) term which we throw away because it does not contribute to the instability. The goal now is to factor out the $\cos\omega t$ and $\sin\omega t$ terms, plug back in $\omega = \omega_0 + \epsilon/2$, omit terms of orders ϵ^2 and higher, and solve the system of equations for the coefficients. The algebra is bad but if we note that to lowest order in ϵ

$$\omega^2 = (\omega_0 + \epsilon/2)^2 = \omega_0^2 + \omega_0\epsilon$$

and persevere we will find that

$$\left(-a\omega_0\epsilon + \gamma b\omega_0 + \frac{1}{2}a\omega_0^2 h\right)\cos\omega t - \left(b\omega_0\epsilon + \gamma a\omega_0 + \frac{1}{2}b\omega_0^2 h\right)\sin\omega t = 0. \tag{3}$$

We have nontrivial solutions when the 2 × 2 system of the coefficients has a 0 determinant. Lets change $\epsilon \to \epsilon_0$ to denote this solution as the threshold frequency mismatch so for any $\epsilon < \epsilon_0$ we will have instability. In matrix form this equation would be, after dropping an overall factor of ω_0 and subbing in $h = 4y_0/\ell$,

$$0 = \begin{vmatrix} -\epsilon_0 + \frac{1}{2}\omega_0 h & \gamma \\ \gamma & \epsilon_0 + \frac{1}{2}\omega_0 h \end{vmatrix} = \left(-\epsilon_0 + \frac{1}{2}\omega_0\right)\left(\epsilon_0 + \frac{1}{2}\omega_0\right) - \gamma^2$$

$$\boxed{\epsilon_0^2 = \left(\frac{\omega_0 y_0}{\ell}\right)^2 - \gamma^2}. \tag{4}$$

Thus, any $\epsilon < \epsilon_0$ will cause instability. However, because there is linear damping the amplitude y_0 must be above a critical value as well. We find this by assuming perfect frequency matching, i.e. by letting $\epsilon \to 0$, and solving for $y_{0,\text{min}}$:

$$\boxed{y_{0,\text{min}} = \frac{\gamma\ell}{\omega_0}}. \tag{5}$$

The physical interpretation is that is we are perfectly on resonance then we MUST drive the oscillator with $y_0 > y_{0,\text{min}}$, otherwise the damping term prevents the parametric instability.

Solution for HW #1

Winter 201
PHYS 200B

4. Let $H(q,p,t) = H_0(q,p) + V(q)\frac{d^2A}{dt^2}$, where $A(t)$ is periodic, with period $\tau << T$. Here T is the motion governed by $H_0 = \frac{p^2}{2m} + V_0(q)$.

a) The mean field are the slow equations with the short time averaged out. First, we find the general equations of motion by Hamilton's equati

$$\dot{p} = -\frac{\partial H}{\partial q} = -\left(\frac{\partial V_0(q)}{\partial q} + \frac{\partial V(q)}{\partial q}\frac{d^2A}{dt^2}\right) \qquad \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$m\ddot{q} = -\frac{\partial V_0}{\partial q} - \frac{\partial V}{\partial q}\frac{d^2A}{dt^2} \qquad p = m\dot{q}$$

$$\dot{p} = m\ddot{q}$$

Now we need to seperate into to two time scales $q = q_s + q_f$, slow and fast respectively, so:

$$m(\ddot{q}_s + \ddot{q}_f) = -\frac{\partial V_0(q_s+q_f)}{\partial q} - \frac{\partial V(q_s+q_f)}{\partial q}\frac{d^2A}{dt^2}$$

The fast motion is supposed to be small in amplitude i.e. a perturbation where $q_f < q_s$, so we expand in small q_f around q_s:

$$m(\ddot{q}_s + \ddot{q}_f) = -\left(\frac{\partial V_0}{\partial q}\bigg|_{q_s} + q_f\frac{\partial^2 V_0}{\partial q}\bigg|_{q_s} + \ldots\right) + $$

$$-\left(\frac{\partial V}{\partial q}\bigg|_{q_s} + q_f\frac{\partial^2 V}{\partial q}\bigg|_{q_s} + \ldots\right)\frac{d^2A}{dt^2}$$

$$\approx \underbrace{-\frac{\partial V_0}{\partial q_s}}_{\substack{\text{slow}\\ \text{(depends on } q_s \\ \text{only)}}} \underbrace{- q_f\frac{\partial^2 V_0}{\partial q_s^2}}_{\substack{\text{fast}\\ (\text{multiplied} \\ \text{by } q_f)}} \underbrace{- \frac{\partial V}{\partial q_s}\cdot\frac{d^2A}{dt^2}}_{\substack{\text{fast}\\ \left(\frac{d^2A}{dt^2}\right)}} \underbrace{- q_f\frac{\partial^2 V}{\partial q_s^2}\frac{d^2A}{dt^2}}_{\substack{\text{slow}\\ \text{(beat term)}}} \leftarrow \begin{array}{l}\text{Now}\\ V = V(q_s)\\ V_0 = V_0(q_s\end{array}$$

Now average over a short time period τ :

$$m(\langle \ddot{q}_s\rangle + \langle \ddot{q}_f\rangle) = -\left(\langle \frac{\partial V_0}{\partial q_s}\rangle + \langle q_f \frac{\partial^2 V_0}{\partial q_s^2}\rangle + \langle \frac{\partial V}{\partial q_s}\frac{d^2A}{dt^2}\rangle + \langle q_f \frac{\partial^2 V}{\partial q_s^2}\frac{d^2A}{dt^2}\rangle\right)$$

For fast equation only fast terms should contribute.

$m\ddot{q}_f = q_f \frac{\partial^2 V_0}{\partial q_s^2} + \frac{\partial V}{\partial q_s}\cdot\frac{d^2A}{dt^2}$, but $\frac{\partial^2 V_0}{\partial q_s^2} \sim \Omega_0^2 = \frac{1}{T^2}$ (small, short frequency)

and $m\ddot{q}_f \sim m\omega^2 q_f = m\left(\frac{1}{\tau^2}\right)q_f >> m\left(\frac{1}{T^2}\right)q_f$, so :

$m\ddot{q}_f = \frac{\partial V}{\partial q_s}\cdot\frac{d^2A}{dt^2} \longrightarrow q_f = \frac{-\partial V}{\partial q_s} A\cdot\frac{1}{m}$ (can't have ~~t~~, t^2; they blow up)

Returning to the slow equation, $V(q_s), V_0(q_s)$ are slowly varying and effectively constant:

$$m\ddot{q}_s + m\cancelto{0}{\langle\ddot{q}_f\rangle} = -\frac{\partial V_0}{\partial q_s} - \frac{\partial^2 V_0}{\partial q_s^2}\cancelto{0}{\langle q_f\rangle} - \frac{\partial V}{\partial q_s}\cancelto{0}{\langle\frac{d^2A}{dt^2}\rangle} + \frac{\partial^2 V}{\partial q_s^2}\langle q_f \frac{d^2A}{dt^2}\rangle$$

The fast terms average to zero over τ :

$$\boxed{m\ddot{q}_s = -\frac{\partial V_0}{\partial q_s} + \frac{1}{m}\frac{\partial V}{\partial q_s}\cdot\frac{\partial^2 V}{\partial q_s}\langle A\frac{d^2A}{dt^2}\rangle = -\frac{\partial V_0}{\partial q} + \frac{1}{2m}\left(\frac{\partial V}{\partial q_s}\right)^2\langle A\frac{d^2A}{dt^2}\rangle}$$

b) The effective Hamiltonian $K(p,q) = H_0(p,q) + \frac{1}{2m}\langle\left(\frac{dA}{dt}\right)^2\rangle\left(\frac{\partial V(q)}{\partial q}\right)^2$ gives:

$$\dot{p} = -\frac{\partial V_0}{\partial q} - \frac{1}{\cancel{2}m}\langle\left(\frac{dA}{dt}\right)^2\rangle\left(\cancel{2}\frac{\partial V}{\partial q}\cdot\frac{\partial^2 V}{\partial q^2}\right), \quad \dot{q} = \frac{p}{m}$$

so $m\ddot{q} = -\frac{\partial V_0}{\partial q} - \frac{1}{m}\frac{\partial V}{\partial q}\frac{\partial^2 V}{\partial q^2}\langle\left(\frac{dA}{dt}\right)^2\rangle$, but

$$\langle A\frac{d^2A}{dt^2}\rangle = \frac{1}{\tau}\int_0^\tau A\frac{d^2A}{dt^2}dt = \frac{1}{\tau}\left(\underbrace{\cancel{A\frac{dA}{dt}\Big|_0^\tau}}_{0} - \int_0^\tau \left(\frac{dA}{dt}\right)\left(\frac{dA}{dt}\right)dt\right) = -\langle\left(\frac{dA}{dt}\right)^2\rangle$$

(periodic)

Substituting this identity into the mean field equation for q_s gives equivalence i.e. the mean field equations can be derived using this effective Hamiltonian $K(p,q)$.

Sheena Patel
Phys200b: Theoretical Mechanics II
Problem Set 1

Problem 5
Consider the asymmetric top, with moments of inertia $I_1 < I_2 < I_3$. Here 1, 2, 3 refer to the principal axes in a frame for which the inertia tensor is diagonal. Using the Euler equations:

a.) Derive the equations of motion for $\Omega_1(t)$, $\Omega_2(t)$, and $\Omega_3(t)$, the angular frequencies associated with axes 1, 2, and 3.

Recall, from rigid body mechanics,

$$\overrightarrow{N}^{\text{ext}} = \left(\frac{\mathrm{d}\overrightarrow{L}}{\mathrm{d}t}\right)_{\text{inertial}} = \left(\frac{\mathrm{d}\overrightarrow{L}}{\mathrm{d}t}\right)_{\text{body}} + \overrightarrow{\Omega} \times \overrightarrow{L} = I\dot{\overrightarrow{\Omega}} + \overrightarrow{\Omega} \times (I\overrightarrow{\Omega})$$

This results in the Euler equations, giving us equations of motion for $\Omega_1(t)$, $\Omega_2(t)$, and $\Omega_3(t)$.

$$I_1\dot{\Omega}_1(t) = (I_2 - I_3)\Omega_2\Omega_3 + N_1^{\text{ext}}$$
$$I_2\dot{\Omega}_2(t) = (I_3 - I_1)\Omega_3\Omega_1 + N_2^{\text{ext}}$$
$$I_3\dot{\Omega}_3(t) = (I_1 - I_2)\Omega_1\Omega_2 + N_3^{\text{ext}}$$

Since there is no external torque on the top, $N^{\text{ext}} = 0$ and

$$\boxed{I_1\dot{\Omega}_1(t) = (I_2 - I_3)\Omega_2\Omega_3}$$
$$\boxed{I_2\dot{\Omega}_2(t) = (I_3 - I_1)\Omega_3\Omega_1}$$
$$\boxed{I_3\dot{\Omega}_3(t) = (I_1 - I_2)\Omega_1\Omega_2}$$

■

b.) Show that if $\Omega_2 \cong \Omega_0$ while Ω_1, Ω_3 start from an infinitesimal perturbation, instability results. Show that $\Omega_1 \cong \Omega_0$ or $\Omega_3 \cong \Omega_0$ is stable.

First, let's consider the case $\Omega_2 \cong \Omega_0$. Let $\overrightarrow{\Omega} = \Omega_0\hat{e}_2 + \overrightarrow{\delta\Omega}$, where $\overrightarrow{\delta\Omega} = (\delta\Omega_1, \delta\Omega_2, \delta\Omega_3)$.

The equations of motion become

$$I_1\delta\dot{\Omega}_1(t) = (I_2 - I_3)\Omega_0\delta\Omega_3 + \mathcal{O}(\delta\Omega_2\delta\Omega_3)$$
$$I_2\delta\dot{\Omega}_2(t) = 0 + \mathcal{O}(\delta\Omega_1\delta\Omega_3)$$
$$I_3\delta\dot{\Omega}_3(t) = (I_1 - I_2)\Omega_0\delta\Omega_1 + \mathcal{O}(\delta\Omega_1\delta\Omega_2)$$

So, to first order in $\delta\Omega_i$,

$$\delta\ddot{\Omega}_1(t) = \Omega_0^2\frac{(I_2 - I_3)(I_1 - I_2)}{I_3 I_1}\delta\Omega_1$$
$$\delta\ddot{\Omega}_3(t) = \Omega_0^2\frac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3}\delta\Omega_3$$

Let $\Omega^2 = \Omega_0^2\frac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3}$.

$$\delta\ddot{\Omega}_1(t) = \Omega^2\delta\Omega_1$$
$$\delta\ddot{\Omega}_3(t) = \Omega^2\delta\Omega_3$$

Since $I_1 < I_2 < I_3$, $I_1 - I_2 < 0$ and $I_2 - I_3 < 0$, so $\Omega^2 > 0$ and $\delta\Omega_1$, $\delta\Omega_3$ have general solution $c_1 e^{\Omega t} + c_2 e^{-\Omega t}$, which increases exponentially with time, resulting in $\boxed{\text{instability}}$. So for $\Omega_2 \cong \Omega_0$, perturbations in $\delta\Omega_1$, $\delta\Omega_3$ result in instability.

■

Next, consider the case $\Omega_1 \cong \Omega_0$. Here, $\overrightarrow{\Omega} = \Omega_0 \hat{e}_1 + \overrightarrow{\delta\Omega}$, where $\overrightarrow{\delta\Omega} = (\delta\Omega_1, \delta\Omega_2, \delta\Omega_3)$.

The equations of motion become

$$I_1 \delta\dot{\Omega}_1(t) = 0 + \mathcal{O}(\delta\Omega_2 \delta\Omega_3)$$

$$I_2 \delta\dot{\Omega}_2(t) = (I_3 - I_1)\Omega_0 \delta\Omega_3 + \mathcal{O}(\delta\Omega_1 \delta\Omega_3)$$

$$I_3 \delta\dot{\Omega}_3(t) = (I_1 - I_2)\Omega_0 \delta\Omega_2 + \mathcal{O}(\delta\Omega_1 \delta\Omega_2)$$

So, to first order in $\delta\Omega_i$,

$$\delta\ddot{\Omega}_2(t) = \Omega_0^2 \frac{(I_3 - I_1)(I_1 - I_2)}{I_3 I_2} \delta\Omega_2 \equiv \Omega^2 \delta\Omega_2$$

$$\delta\ddot{\Omega}_3(t) = \Omega_0^2 \frac{(I_1 - I_2)(I_3 - I_1)}{I_2 I_3} \delta\Omega_3 \equiv \Omega^2 \delta\Omega_3$$

Now, $\Omega^2 < 0$, so $\delta\Omega_2$, $\delta\Omega_3$ have general solution $c_1 e^{i|\Omega|t} + c_2 e^{-i|\Omega|t}$, which are oscillating solutions and result in $\boxed{\text{stable motion}}$.

■

Similarly, consider the case $\Omega_3 \cong \Omega_0$. Here, $\overrightarrow{\Omega} = \Omega_0 \hat{e}_3 + \overrightarrow{\delta\Omega}$, where $\overrightarrow{\delta\Omega} = (\delta\Omega_1, \delta\Omega_2, \delta\Omega_3)$.

The equations of motion become

$$I_1 \delta\dot{\Omega}_1(t) = (I_2 - I_3)\Omega_0 \delta\Omega_2 + \mathcal{O}(\delta\Omega_2 \delta\Omega_3)$$

$$I_2 \delta\dot{\Omega}_2(t) = (I_3 - I_1)\Omega_0 \delta\Omega_1 + \mathcal{O}(\delta\Omega_1 \delta\Omega_3)$$

$$I_3 \delta\dot{\Omega}_3(t) = 0 + \mathcal{O}(\delta\Omega_1 \delta\Omega_2)$$

To first order in $\delta\Omega_i$,

$$\delta\ddot{\Omega}_1(t) = \Omega_0^2 \frac{(I_2 - I_3)(I_3 - I_1)}{I_2 I_1} \delta\Omega_1 \equiv \Omega^2 \delta\Omega_1$$

$$\delta\ddot{\Omega}_2(t) = \Omega_0^2 \frac{(I_3 - I_1)(I_2 - I_3)}{I_1 I_2} \delta\Omega_2 \equiv \Omega^2 \delta\Omega_2$$

Just as in the last case, $\Omega^2 < 0$, so $\delta\Omega_1$, $\delta\Omega_2$ have general solution $c_1 e^{i|\Omega|t} + c_2 e^{-i|\Omega|t}$, which are oscillating solutions and result in $\boxed{\text{stable motion}}$.

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c.) What are the two conserved quantities which constrain the evolution in b.)?

The two conserved quantities are energy and the square of the angular momentum.

$$\boxed{E = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)}$$

$$\boxed{|\overrightarrow{L}|^2 = I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_3^2}$$

■